Homework 4

Problem (6 points). In the Newtonian limit (nonrelativistic speeds, weak gravitational field), the first-order continuity and momentum conservation equations of a fluid are:

$$\frac{\partial \delta\rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\bar{\rho}\boldsymbol{u}), \tag{1}$$

$$\bar{\rho}\frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{\nabla}\delta P - \bar{\rho}\boldsymbol{\nabla}\delta\psi, \tag{2}$$

where $\rho = \bar{\rho} + \delta\rho$ is the density, $\boldsymbol{u}$ the bulk velocity, P is the pressure and ψ the gravitational potential. If the gravitational potential is entirely due to the fluid considered, then ψ is related to the density through the Poisson equation:

$$\nabla^2\delta\psi = 4\pi G\delta\rho. \tag{3}$$

Finally, perturbations in pressure and density are related as follows:

$$\delta P = \frac{\partial P}{\partial \rho}\delta\rho \equiv c_s^2\delta\rho, \tag{4}$$

where c_s is the speed of sound.

i): combine the equations above[1] to obtain the following differential equation in $\delta\rho$:

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2 - 4\pi G\bar{\rho}\right)\delta\rho = 0. \tag{5}$$

ii): in Fourier space, the above equation becomes

$$\left(\frac{\partial^2}{\partial t^2} + c_s^2k^2 - 4\pi G\bar{\rho}\right)\tilde{\delta\rho}(\boldsymbol{k}, t) = 0. \tag{6}$$

Look for solutions of the form

$$\tilde{\delta\rho}(\boldsymbol{k}, t) = Ae^{i\boldsymbol{k}\boldsymbol{x}}e^{i\omega t}. \tag{7}$$

Find the relation between the frequency ω and the wavevector $\boldsymbol{k}$ (the dispersion relation). Discuss the qualitative behavior of solutions with different values of k.

[1] Take the gradient of Equation 2