Homework 5

Problem (8 points). The continuity and momentum conservation equations for cold dark matter in an expanding universe with metric perturbations are

$$\frac{\partial \rho_c}{\partial t} + \frac{1}{a}\boldsymbol{\nabla}\cdot(\rho_c \boldsymbol{u}_c) + 3(H + \dot{\phi})\rho_c = 0, \tag{1}$$

$$\frac{\partial(\rho_c \boldsymbol{u}_c)}{\partial t} + 4H\rho_c \boldsymbol{u}_c + \frac{1}{a}\rho_c \boldsymbol{\nabla}\psi = 0. \tag{2}$$

i): expressing the density as $\rho_c = \bar{\rho}_c(1+\delta_c)$ and keeping only the first-order terms $(\delta_c, u_c, \phi, \psi)$, show[1] that the corresponding first-order equations are

$$\frac{\partial \delta_c}{\partial t} + \frac{1}{a}\boldsymbol{\nabla}\cdot(\bar{\rho}\boldsymbol{u}) + 3\dot{\phi} = 0, \tag{3}$$

$$\frac{\partial \boldsymbol{u}_c}{\partial t} + H\boldsymbol{u}_c + \frac{1}{a}\boldsymbol{\nabla}\psi = 0. \tag{4}$$

In a *matter-dominated universe*, in the limit of *small scales* (compared to $1/(aH)$), and assuming *no anisotropic stress* ($\Theta_2 = 0$) the time-time component of Einstein equations becomes:

$$\boldsymbol{\nabla}^2\psi \approx 4\pi G a^2 \bar{\rho}_c \delta_c. \tag{5}$$

Furthermore, terms dependent on ϕ in Equation (3) can be neglected.

ii): Following a similar approach to that of HW4, obtain the following differential equation

$$\frac{\partial^2 \delta_c}{\partial t^2} + 2H\frac{\partial \delta_c}{\partial t} - 4\pi G\bar{\rho}_c\delta_c = 0. \tag{6}$$

iii): Show that, in a matter-dominated universe, the equation above is equivalent to

$$\frac{\partial^2 \delta_c}{\partial t^2} + \frac{4}{3t}\frac{\partial \delta_c}{\partial t} - \frac{2}{3t^2}\delta_c = 0. \tag{7}$$

iv): Look for solutions of the form $\delta_c(t) \propto t^p$ to the above differential equation. Discuss the solutions.

[1] First derive the zero-order equation and use it to simplify the first-order one.